ELSEVIER Discrete Mathematics 174 (1997) 383-386

DISCRETE MATHEMATICS

Monohedrally knotted tilings of the 3-space

Joseph Zaks

Department of Mathematics, University of Haifa, Mt. Carmel, 31905 Haifa, Israel

Received 2 November 1994

Abstract

We present our simplified version of a result we heard recently in the Dortmund meeting on Discrete Mathematics, July, 1994, given independently by P. Schmidt and by W. Kuperberg.

A tile M is said to tile E^3 monohedrally, if E^3 can be tiled by congruent copies of M (see [2]). The following theorem was attributed to P. McMullen (unpublished example) by P. Schmidt and by Kuperberg [1].

Theorem 1. *Let K be any polyhedral knot in E^3; for any $\varepsilon > 0$ there exists a tile M which is isotopic to an ε-neighbourhood of K, such that M monohedrally tiles E^3.*

Actually, K need not be a knot; we have a simple construction to establish the following version of Theorem 1.

Theorem 2. *Let G be any connected graph, and let $f: G \to E^3$ be any piece-wise linear embedding of G into E^3, and for any $\varepsilon > 0$, there exists a tile M which is isotopic to an ε-neighbourhood of a set H, isotopic to $f(G)$, such that M monohedrally tiles E^3.*

Theorem 2 is established in two steps: first we get a suitable isotopic version of $f(G)$, then we construct the tile M.

A 3-pages is a set homeomorphic to the Cartesian product of a Y-shape and a segment; the *spine* is the product of the center-point of the Y-shape and the segment. We need the following lemma.

Lemma. *For every connected graph G and for every piece-wise linear embedding $f: G \to E^3$, $f(G)$ is isotopic to a graph H, embedded in a 3-pages.*

PII S0012-365X(96)00351-2

Proof of the Lemma. Let G be a connected graph, and let f be a piece-wise linear embedding $f: G \to E^3$. Isotopically change $f(G)$ so as to try and bring all of $f(G)$ into a plane. Of course, this will usually be impossible, since $f(G)$ can be knotted and far from being a planar set. However, it can be isotopically stretched to be almost in a plane, except for over-passes and under-passes. Isotopically twist it suitably, so that all the passes are on a line, and all the under-passes become over-passes (along that line). Most of the image of G is now in a plane (= two pages), except for a discrete collection of over-passes; all of these overpasses can be accommodated in the third page; thus $f(G)$ is isotopic to a subset of a 3-pages as required. □

We present our main result.

Proof of Theorem 2. Let $f: G \to E^3$ be an embedding of a connected graph G. Using the Lemma, let H be an isotopic image of $f(G)$ which lies in a 3-pages; in fact, H can even be a subset of a 4-pages.

To get the tile M, we proceed as follows.

Start with the regular tiling of E^3 by lattice-unit cubes: $\{(x, y, z) \mid a \leqslant x \leqslant a + 1, b \leqslant y \leqslant b + 1, c \leqslant z \leqslant c + 1, a, b, c \in \mathbb{Z}\}$. Translate every other layer of cubes (for all odd a) by the vector $(0, \frac{1}{2}, \frac{1}{2})$. Thus, each one of the mid-points $(a, b + \frac{1}{2}, c + \frac{1}{2})$ of the facets of the cubes meets four other cubes, as shown in Fig. 1.

Without loss of generality, let H be a subset of the 4-pages having its spine S on the segment from $(a, b + \frac{1}{2}, c + \frac{1}{2})$ to $(a + \frac{1}{2}, b + \frac{1}{2}, c + \frac{1}{2})$, and let its four pages be N, S, E and W (see Fig. 2).

For every edge of H that lies in the spine, add to each cube a small cylinder of radius ε along the segment of the form $(a + n\delta, b + \frac{1}{2}, c + \frac{1}{2}) - (a + (n + 1)\delta, b + \frac{1}{2}, c + \frac{1}{2})$, for a suitable n and some small δ; in addition, four halves of such a cylinder are deleted along the four (parallel) edges of the cube, which are in the direction $\boldsymbol{i}$, as shown in Fig. 3.

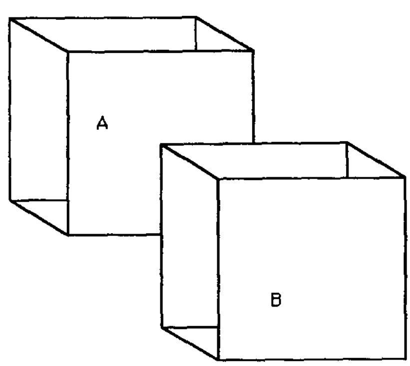

Fig. 1.

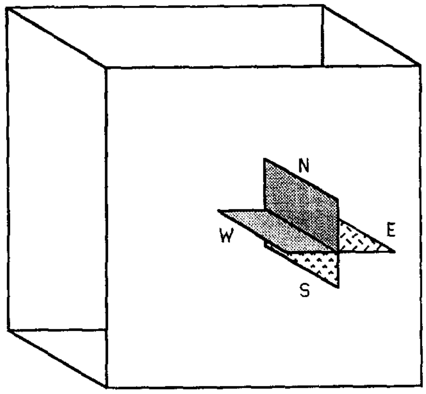

Fig. 2.

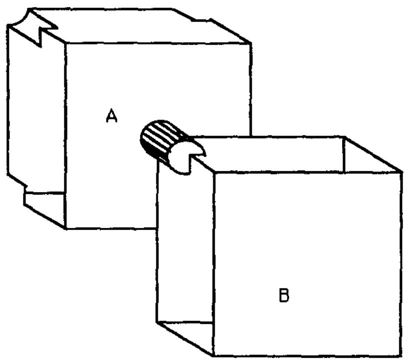

Fig. 3.

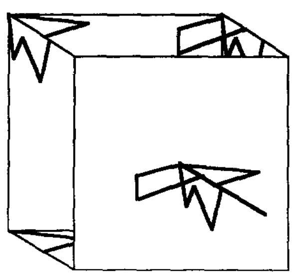

Fig. 4.

For every edge of H which lies in one of the pages N, S, E or W, add a suitable cylinder to the cube, a cylinder having its center in that page, and delete two imprints of this cylinder from two opposite faces of the cube, which are parallel to that page, as shown in Fig. 4.

Observe that the two half-imprints allow for the insertion of a suitable cylinder in two other cells. At the final step, each cell is isotopic to an ε-neighborhood of H.

This completes the demonstration of our construction.

References

[1] W. Kuperberg, Knotted lattice-like space fillers, manuscript, 1994.
[2] B. Grünbaum and G. Shephard, Tilings and Patterns (Freeman, New York, 1987).